Phys 410
Spring 2013
Lecture #35 Summary
19 April, 2013

We continued the discussion of coupled oscillators by considering two masses on a friction-less surface, with 3 springs between them. The left mass (m_1) is connected to the left wall by a spring of spring constant k_1, while the other mass (m_2) is connected to the right wall by a spring of spring constant k_3. The two masses are also directly connected to each other by a third spring characterized by k_2. In the absence of spring 2, the two masses would oscillate independently at their own natural frequencies. However, with the coupling, they will have a new type of motion characterized by 'normal modes.'

We wrote down the Lagrangian of the system and found that Lagrange's equations yield a pair of coupled second-order linear differential equations: $-(k_1 + k_2)x_1 + k_2 x_2 = m_1 \ddot{x}_1$, and $k_2 x_1 - (k_2 + k_3)x_2 = m_2 \ddot{x}_2$. These equations can be summarized in an elegant 2x2 matrix equation: $\bar{\bar{M}}\ddot{\vec{x}} = -\bar{\bar{K}}\vec{x}$, where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the vector of unknowns, $\bar{\bar{M}} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ is the "mass matrix", and $\bar{\bar{K}} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}$ is the "spring constant matrix". This equation is a generalization of the mass on a spring equation. In fact it reduces to two un-coupled mass/spring equations when $k_2 = 0$.

Although the two un-coupled masses would oscillate on their own at different frequencies, we are going to try an ansatz in which both masses oscillate together at a single frequency. We use the complex form, which worked so well for the single harmonic oscillator, but now generalized to 2 oscillators: $\vec{x}(t) = Re\left[\vec{C}e^{i\omega t}\right]$, where $\vec{C} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$, and C_1 and C_2 are complex constants. Putting this into the matrix equation yields $\left(\bar{\bar{K}} - \omega^2\bar{\bar{M}}\right)\vec{C} = 0$. This is similar to, but not exactly, an eigenvalue problem (the two different values of mass prevents it from being an eigenvalue problem). Nevertheless, we can still use the formalism of linear algebra to solve this problem. To get a non-trivial solution for $\vec{C}$, we demand that $det\left(\bar{\bar{K}} - \omega^2\bar{\bar{M}}\right) = 0$. This yields a quadratic equation for ω^2, with two solutions.

We then specialized to the case of equal masses (m) and equal spring constants (k). The quadratic equation then yields two normal mode frequencies: $\omega_1 = \sqrt{k/m}$, and $\omega_2 = \sqrt{3k/m}$. The corresponding normal modes were found to be $x_1 = x_2 = A\cos(\omega_1 t - \delta_1)$ for ω_1 (this is the 'sloshing mode') and $x_1 = -x_2 = A\cos(\omega_2 t - \delta_2)$ for ω_2 (this is the 'beating mode'). The general solution is a linear combination of these two normal modes with arbitrary weighting constants.

The choice of two new coordinates, so-called normal coordinates, would have diagonalized the $\overline{\overline{K}}$ matrix from the get-go. In this case the normal coordinates are $\xi_1 = \frac{1}{2}(x_1 + x_2)$, and $\xi_2 = \frac{1}{2}(x_1 - x_2)$. Each obeys an un-coupled equation of motion and the two 'oscillators' have normal mode frequencies of $\omega_1 = \sqrt{k/m}$, and $\omega_2 = \sqrt{3k/m}$.

We then considered another coupled oscillator problem – the double pendulum. We wrote down the Lagrangian, which turned out to be quite complicated. It leads to nonlinear equations of motion – as is well known for the single pendulum. To avoid this problem (which we will deal with later), we made a "small oscillations" approximation for the double pendulum. In this approximation we take ϕ_1, ϕ_2, $\dot{\phi}_1$, and $\dot{\phi}_2$ to be small, and only keep terms up to second order in these quantities. We then did a Taylor series expansion for the kinetic energy and potential energy to arrive at an approximate Lagrangian of the form: $\mathcal{L} = \frac{1}{2}(m_1 + m_2)\left(L_1\dot{\phi}_1\right)^2 + m_2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 + \frac{1}{2} m_2 \left(L_2 \dot{\phi}_2\right)^2 - \frac{(m_1+m_2) g L_1 \phi_1^2}{2} - \frac{m_2 g L_2 \phi_2^2}{2}$. Both the kinetic energy and the potential energy are homogeneous quadratic functions.